Linear Programming

Lecture 2: Introduction to Linear Programming

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- The function to be minimized or maximized is called the *objective function*.
- The set of alternatives is called the feasible region (or constraint region).
- In this course, the feasible region is always taken to be a subset of $\mathbb{R}^n$ (real n-dimensional space) and the objective function is a function from $\mathbb{R}^n$ to $\mathbb{R}$.

What is linear programming (LP)?

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A *linear program* is an optimization problem in finitely many variables having a linear objective function and a constraint region determined by a finite number of linear equality and/or inequality constraints.

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that are linear equality and/or linear inequality constraints.

- A linear function of the variables $x_1, x_2, \ldots, x_n$ is any function f of the form

$$f(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

for fixed $c_i \in \mathbb{R}$ $i = 1, \ldots, n$.

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- A linear equality constraint is any equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \alpha,$$

  where $\alpha, a_1, a_2, \ldots, a_n \in \mathbb{R}$.

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- A linear inequality constraint is any inequality of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq \alpha,$$

or

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq \alpha,$$

where $\alpha, a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Compact Representation

$$
\begin{aligned}
\text{maximize} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} \quad & a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n \leq \alpha_i \qquad i = 1, \ldots, s \\
& b_{i1}x_i + b_{i2}x_2 + \cdots + b_{in}x_n = \beta_i \qquad i = 1, \ldots, r.
\end{aligned}
$$

Vector Inequalities: Componentwise

Let $x, y \in \mathbb{R}^n$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

We write $x \leq y$ if and only if

$$x_i \leq y_i, \;\; i = 1, 2, \ldots, n \; .$$

Matrix Notation

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = c^Tx$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix Notation

$$a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n \leq \alpha_i \;\; i = 1, \ldots, s$$

$$\Longleftrightarrow$$

$$Ax \leq a$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \ldots & a_{sn} \end{bmatrix} \qquad a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_s \end{bmatrix}$$

Matrix Notation

$$b_{i1}x_i + b_{i2}x_2 + \cdots + b_{in}x_n = \beta_i \quad i = 1, \ldots, r$$

$$\Longleftrightarrow$$

$$Bx = b$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \ldots & b_{rn} \end{bmatrix} \qquad b = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_r \end{bmatrix}$$

LP's Matrix Notation

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq a \text{ and } Bx = b \end{array}$$

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Applications of Linear Programing

A very short list:

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Example: Plastic Cup Factory

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

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Model this problem as an LP.

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4. Determine the *implicit constraints* and write them as a linear equation/inequality in the decision variables.

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This last point cannot be over emphasized. Even the most experienced modelers occasionally fall into this trap since such assumptions can enter in very subtle ways.

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$C =$ number of cases of champagne glasses produced daily

Objective Function

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Maximize Profit:

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Maximize Profit: Profit = Revenue − Costs

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Maximize Profit: $\text{Profit} = \text{Revenue} - \text{Costs}$

$$\text{Profit} = 25B + 20C$$

Explicit Constraints

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Resin: $20B + 12C \leq 1800$

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Labor: $B/15 + C/15 \leq 8$

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Implicit Constraints:
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The Plastic Cup Factory LP Model

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Never be afraid to add more decision variables either to clarify the model or to improve its flexibility. Modern LP software easily solves problems with thousands of variables on a laptop, tens of thousands of variables on a server, or even tens of millions of variables on specialized hardware and networks. It is more important to get a correct, easily interpretable, and flexible model then to provide a compact minimalist model.

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LP model solutions found in many texts fall into the trap of trying to provide the most compact minimalist model with the fewest possible variables and constraints. **Do not repeat this error in developing your own models.**

Graphical Solution of 2D LPs

We now graphically solve the LP model for the Plastic Cup Factory problem.

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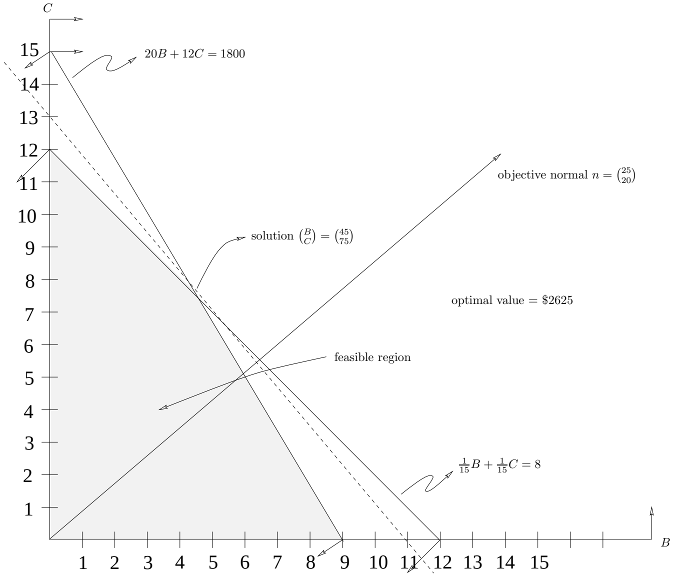

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Step 1: Graph each of the linear constraints indicating on which side of the constraint the feasible region must lie with an arrow. Don't forget the implicit constraints!

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Step 4: Place a straight-edge perpendicular to the gradient vector.

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Step 2: Shade in the feasible region.

Step 3: Draw the gradient vector of the objective function.

Step 4: Place a straight-edge perpendicular to the gradient vector.
Move the straight-edge in the direction of the gradient vector for maximization (or in the opposite direction for minimization).

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Step 3: Draw the gradient vector of the objective function.

Step 4: Place a straight-edge perpendicular to the gradient vector.
Move the straight-edge in the direction of the gradient vector for maximization (or in the opposite direction for minimization).
Move to the last point for which the straight-edge intersects the feasible region.

Recap: Graphical Solution of 2D LPs

Step 1: Graph each of the linear constraints indicating on which side of the constraint the feasible region must lie with an arrow. Don't forget the implicit constraints!

Step 2: Shade in the feasible region.

Step 3: Draw the gradient vector of the objective function.

Step 4: Place a straight-edge perpendicular to the gradient vector.
Move the straight-edge in the direction of the gradient vector for maximization (or in the opposite direction for minimization).
Move to the last point for which the straight-edge intersects the feasible region.

Step 5: The set of points of intersection between the straight-edge and the feasible region is the set of solutions to the LP. Compute these points precisely along with the associated optimal value.

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Problems with the input data for *real world* LPs.

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We need to be able to study how the optimal value and solution change as the problem input data change.

The Optimal Value Function

$$
\begin{aligned}
v(\epsilon_1, \epsilon_2) = \quad \text{maximize} \quad & 25B + 20C \\
\text{subject to} \quad & 20B + 12C \leq 1800 + \epsilon_1 \\
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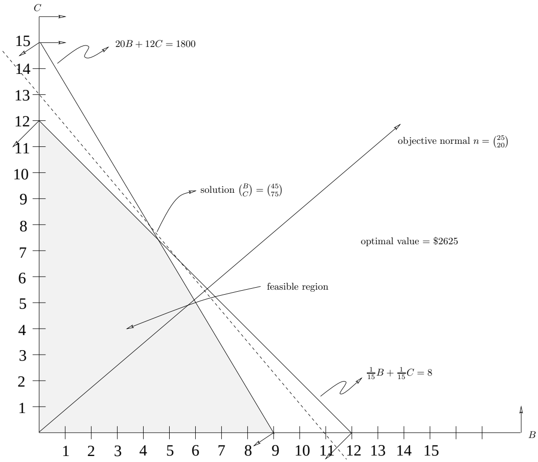
Vertex Solutions



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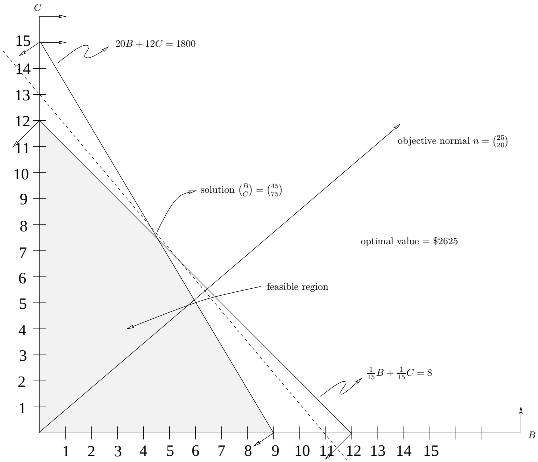

The optimal solution lies at a "corner point" or "vertex" of the feasible region.

Vertex Solutions

Conjecture: For a small range of perturbations to the resources, the vertex associated with the current optimal solution moves but remains optimal.

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The conjecture implies that the solution to the perturbed LP lies at the intersection of the two lines $20B + 12C = 1800 + \epsilon_1$ and $\frac{1}{15}B + \frac{1}{15}C = 8 + \epsilon_2$ for small values of ϵ_1 and ϵ_2; namely

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It can be verified by direct computation that this indeed yields the optimal solution for small values of ϵ_1 and ϵ_2.

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The components of the gradient are called the marginal values for the resources.

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Linear theory of production

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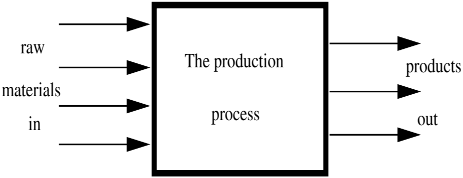

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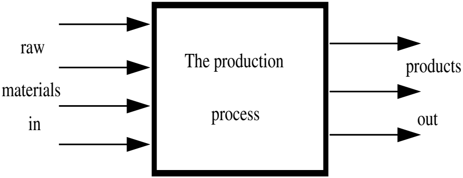

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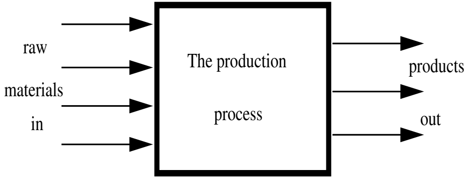

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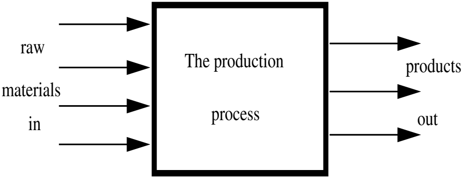

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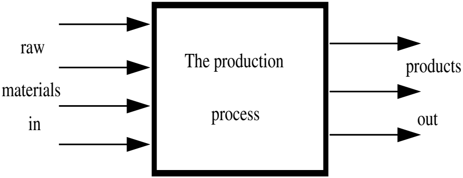

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Solution: The marginal values!

$$
\nabla v(\epsilon_1, \epsilon_2) = \left[\begin{array}{c} 5/8 \\ 375/2 \end{array} \right]
$$

Hidden Hand of the Market Place: Duality

In the market place there is competition for raw materials, or the inputs to production. This collective competition is the *hidden hand* that sets the price for goods in the market place.

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How can we model this mathematically?

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A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is $25 while the profit on a case of champagne glasses is $20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

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These price increases should wipe out the per unit profitability for cases of both beer mugs and champagne glasses.

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production cost increase $\geq$ current profit

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Let us compare this LP with the original LP.

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Linear Programming Duality

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s.t. $20B + 12C \leq 1800$

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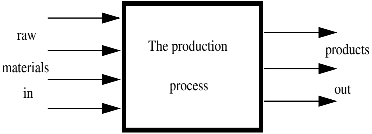

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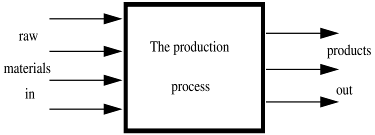

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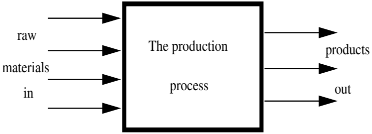

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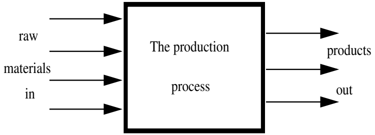

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And indeed, they are the solution!

Linear Programming Duality: Matrix Notation

	$\mathcal{P}$			$\mathcal{D}$	
Primal:	max	$c^T x$	**Dual:**	min	$b^T y$
	s.t.	$Ax \leq b$		s.t.	$A^T y \geq c$
		$0 \leq x$			$0 \leq y$

The Weak Duality Theorem of Linear Programming

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Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible.

Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

The Weak Duality Theorem of Linear Programming

Proof:

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$$c^T x = \sum_{j=1}^{n} c_j x_j$$

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\begin{aligned}
c^T x &= \sum_{j=1}^{n} c_j x_j \\
&\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i\right) x_j \quad \left[0 \leq x_j,\ c_j \leq \sum_{i=1}^{m} a_{ij} y_i \Rightarrow\ c_j x_j \leq \left(\sum_{i=1}^{m} a_{ij} y_i\right) x_j\right]
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Test the WDT on the Plastic Cup Factory

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$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

$$\text{Marginal Values} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$$

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Dual:

$$\begin{array}{lllll} \text{min} & 1800y_1 + 8y_2 & & & \\ \text{s.t.} & 20y_1 & + & (1/15)y_2 & \geq 25 \\ & 12y_1 & + & (1/15)y_2 & \geq 20 \\ & 0 \leq y_1, y_2 & & & \end{array}$$

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Equivalence of primal-dual objectives (WDT):

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Equivalence of primal-dual objectives (WDT):

$$c^T x = 25 \cdot 45 + 20 \cdot 75 = 2625$$

Test the WDT on the Plastic Cup Factory

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Dual:

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Dual feasibility of the marginal values:

$$0 \leq \frac{5}{8}, \; 0 \leq \frac{375}{2}, \quad 20 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 25, \quad 12 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 20$$

Equivalence of primal-dual objectives (WDT):

$$c^T x = 25 \cdot 45 + 20 \cdot 75 = 2625 = 1800 \cdot \frac{5}{8} + 8 \cdot \frac{375}{2} = b^T y$$

What the Weak Duality Theorem Does NOT Say

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Example:

$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & - & x_2 & & \\ & x_1 & - & x_2 & \leq & 1 \\ & -x_1 & + & x_2 & \leq & -2 \\ & 0 & \leq & x_1, & x_2 & \end{array}$$